INVARIANT METRICS AND DISTANCES ON GENERALIZED NEIL PARABOLAS

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Abstract. We present the Carathéodory-Reiffen metric and the inner Carathéodory distance on generalized parabolas. It turns out that on such parabolas the Carathéodory distance is not inner.

1. Introduction and results

In the survey paper [3] the authors had asked for an effective formula for the Carathéodory distance $c_{A_{2,3}}$ on the Neil parabola $A_{2,3}$ (in the bidisc). In a recent paper, such a formula was presented by G. Knese. To repeat the main result of [4] recall that the Neil parabola is given by $A_{2,3} := \{(z,w) \in \mathbb{D}^2 : z^2 = w^3\}$, where $\mathbb{D}$ denotes the open unit disc in the complex plane. Then there is the natural parametrization $p_{2,3} : \mathbb{D} \longrightarrow A_{2,3}$, $p_{2,3}(\lambda) := (\lambda^3, \lambda^2)$. Moreover, let ρ denote the Poincaré distance of the unit disc. Recall that $\rho(\lambda,\mu) := \frac{1}{2}\log\frac{1+m_{\mathbb{D}}(\lambda,\mu)}{1-m_{\mathbb{D}}(\lambda,\mu)}$, where $m_{\mathbb{D}}(\lambda,\mu) := |\frac{\lambda-\mu}{1-\lambda\overline{\mu}}|$, $\lambda,\mu \in \mathbb{D}$.

Let $\lambda,\mu \in \mathbb{D}$. Then Knese's result is the following one:

$$c_{A_{2,3}}(p_{2,3}(\lambda), p_{2,3}(\mu)) = \begin{cases} \rho(\lambda^2,\mu^2) & \text{if } |\alpha_0| \geq 1 \\ \rho\left(\lambda^2\frac{\alpha_0-\lambda}{1-\overline{\alpha}_0\lambda}, \mu^2\frac{\alpha_0-\mu}{1-\overline{\alpha}_0\mu}\right) & \text{if } |\alpha_0| < 1 \end{cases},$$

where $\alpha_0 := \alpha_0(\lambda,\mu) := \frac{1}{2}(\lambda + \frac{1}{\lambda} + \mu + \frac{1}{\mu})$. In the case when $\lambda\mu = 0$ the formula should be read as in the case $|\alpha_0| \geq 1$.

Observe that if λ and μ have a non-obtuse angle, i.e., $\mathrm{Re}(\lambda\overline{\mu}) \geq 0$, then $|\alpha_0(\lambda,\mu)| > 1$ (compare with Corollary 2).

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Moreover, in [4] the formula for the Carathéodory-Reiffen pseudometric $\gamma_{A_{2,3}}$ is given as:

$$\gamma_{A_{2,3}}((a,b);X)=\begin{cases}|X_2| & \text{if } a=b=0, |X_2|\geq 2|X_1|\\ |X_1| & \text{if } a=b=0, |X_2|<2|X_1|\\ \frac{2|\lambda b|}{1-|b|^2} & \text{if } (a,b)\neq(0,0), X=\lambda(3a,2b),\ \lambda\in\mathbb{C}\end{cases},$$

where $(a,b)\in A_{2,3}$ and $X\in T_{(a,b)}A_{2,3}:=$ the tangent space in (a,b) at $A_{2,3}$.

We point out that these are the first effective formulas for the Carathéodory distance and the Carathéodory-Reiffen pseudodistance of a non-trivial complex space.

In this paper we will discuss more general Neil parabolas, namely the spaces

$$A_{m,n}:=\{(z,w)\in\mathbb{D}^2: z^m=w^n\},\ m,n\in\mathbb{N},\ m\leq n,\ \text{relatively prime.}$$

For short, we will call $A_{m,n}$ the (m,n)-*parabola*. As in the case of the classical Neil parabola we have the following globally bijective holomorphic parametrization of $A_{m,n}$, namely

$$p_{m,n}:\mathbb{D}\longrightarrow A_{m,n},\quad p_{m,n}(\lambda):=(\lambda^n,\lambda^m),\ \lambda\in\mathbb{D}.$$

Observe that $q_{m,n}:=p_{m,n}^{-1}:A_{m,n}\longrightarrow\mathbb{D}$ is given outside of the origin by $q_{m,n}(z,w)=z^kw^l$ where $k,l\in\mathbb{Z}$ are such that $kn+lm=1$; moreover, $q_{m,n}(0,0)=0$. It is clear that $q_{m,n}$ is continuous on $A_{m,n}$ and holomorphic outside of the origin.

We will study the Carathéodory and the Kobayashi distances and also the Carathéodory-Reiffen and the Kobayashi-Royden pseudometrics of $A_{m,n}$. So let us recall the objects we will deal with in this paper:

$$m_{A_{m,n}}(\zeta,\eta):=\sup\{m_{\mathbb{D}}(f(\zeta),f(\eta)):f\in\mathcal{O}(A_{m,n},\mathbb{D})\},\quad \zeta,\eta\in A_{m,n},$$

where $\mathcal{O}(A_{m,n},\mathbb{D})$ denotes the family of holomorphic functions on $A_{m,n}$, i.e., the family of those functions on $A_{m,n}$ that are locally restriction of holomorphic functions on an open set in $\mathbb{C}^2$.

Observe that the Carathéodory distance $c_{A_{m,n}}$ is given by $c_{A_{m,n}}(\zeta,\eta)$ $=\tanh^{-1} m_{A_{m,n}}(\zeta,\eta)$; moreover, $c_{\mathbb{D}}=\rho$.

So, we have to study holomorphic function on the (m,n)-parabola. Recall that there is the following bijection of $\mathcal{O}(A_{m,n},\mathbb{D})$ and a part $\mathcal{O}_{m,n}(\mathbb{D})$ of $\mathcal{O}(\mathbb{D},\mathbb{D}))$, where

$$\mathcal{O}_{m,n}(\mathbb{D}):=\{h\in\mathcal{O}(\mathbb{D},\mathbb{D}):h^{(s)}(0)=0,s\in S_{m,n}\}$$

and $S_{m,n}:=\{s\in\mathbb{N}:s\notin\mathbb{Z}_+m+\mathbb{Z}_+n\}$ (recall that $S_{1,n}=\emptyset$ and if $m\geq 2$, then $\max\limits_{s\in S_{m,n}} s=nm-m-n$). To be precise, if $f\in\mathcal{O}(A_{m,n},\mathbb{D})$

then $f\circ p_{m,n}\in\mathcal{O}_{m,n}(\mathbb{D})$, and conversely, if $h\in\mathcal{O}_{m,n}(\mathbb{D})$ then $h\circ q_{m,n}\in\mathcal{O}(A_{m,n},\mathbb{D})$.

From this consideration it follows that there is the following description of the Caratheódory distance on $A_{m,n}$:

$$\begin{aligned}
&m_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))=\max\{m_{\mathbb{D}}(h(\lambda),h(\mu)):h\in\mathcal{O}_{m,n}(\mathbb{D})\}\\
&=\max\{m_{\mathbb{D}}(h(\lambda),h(\mu)):h\in\mathcal{O}_{m,n}(\mathbb{D}),h(0)=0\}\\
&=\max\{m_{\mathbb{D}}(\lambda^m h(\lambda),\mu^m h(\mu)):h\in\mathcal{O}(\mathbb{D},\overline{\mathbb{D}}),\\
&\qquad h^{(j)}(0)=0,j+m\in S_{m,n}\},\quad \lambda,\mu\in\mathbb{D}.
\end{aligned}$$

We like to mention that the calculation of the Carathéodory distance of a generalized Neil parabola may be read as the following interpolation problem for holomorphic functions on the unit disc. Let λ,μ be as above and let $\zeta,\eta\in\mathbb{D}$. Then there exists an $h\in\mathcal{O}_{m,n}(\mathbb{D})$ with $h(\lambda)=\zeta,h(\mu)=\eta$ if and only if $m_{\mathbb{D}}(\zeta,\eta)\leq m_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))$. Note that $m_{A_{1,n}}(p_{1,n}(\lambda),p_{1,n}(\mu))=m_{\mathbb{D}}(\lambda,\mu)$.

From the case of domains in $\mathbb{C}^n$ it is well known that the Carathéodory distance need not to be an inner distance (see [2]). In the case of a generalized Neil parabola it turns out that the Carathéodory distance is an inner distance if and only if $m=1$.

Recall that the associated inner distance is given by

$$\begin{aligned}
c^i_{A_{m,n}}(\zeta,\eta):=\inf\{L_{c_{A_{n,m}}}(\alpha):\alpha\text{ is a }\|\cdot\|-\text{rectifiable curve in}\\
A_{m,n}\text{ connecting }\zeta,\eta\},\zeta,\eta\in A_{m,n},
\end{aligned}$$

where $L_{c_{A_{m,n}}}$ denotes the $c_{A_{m,n}}$-length. Obviously, $c_{A_{m,n}}\leq c^i_{A_{m,n}}$. Then we have the following result for the inner distance.

Theorem 1. *Let $\lambda,\mu\in\mathbb{D}$. Then*

$$c^i_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))
=\begin{cases}c_{\mathbb{D}}(\lambda^m,\mu^m) & \text{if } \operatorname{Re}(\lambda\overline{\mu})\geq\cos(\pi/m)|\lambda\mu|\\ c_{\mathbb{D}}(\lambda^m,0)+c_{\mathbb{D}}(0,\mu^m) & \text{if otherwise}\end{cases}.$$

Moreover, there is the following comparison result between the Carathéodory distance and its associated inner one.

Corollary 2. *Let $\lambda,\mu\in\mathbb{D}$.*

(a) *If $\operatorname{Re}(\lambda\overline{\mu})\geq\cos(\pi/m)|\lambda\mu|$, then*

$$c^i_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))=c_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu)).$$

(b) *If $\operatorname{Re}(\lambda\overline{\mu})<\cos(\pi/m)|\lambda\mu|$, then*

$$c^i_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))=c_{A_{m,n}}((p_{m,n}(\lambda),p_{m,n}(\mu))\ \textit{iff}\ (\lambda\overline{\mu})^m<0.$$

Thus, the following conditions are equivalent.

- $c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$;
- $c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c_{\mathbb{D}}(\lambda^m, \mu^m)$;
- $\mathrm{Re}(\lambda\overline{\mu}) \geq \cos(\pi/m)|\lambda\mu|$ *or* $(\lambda\overline{\mu})^m < 0$.

In particular, $c_{A_{m,n}}$ *is not inner if* $m > 1$.

Observe that the condition $\mathrm{Re}(\lambda\overline{\mu}) \geq \cos(\pi/m)|\lambda\mu|$ in these results means geometrically that μ lies inside an angular sector around λ of opening angle equal π/m (compare with Knese's result from above). Moreover, opposite to the $A_{2,3}$-case the new area $(\lambda\overline{\mu})^m < 0$ (i.e., the "rays" on which the angle between λ and μ equals to $\frac{(2j-1)\pi}{m}$, $j = 2, \dots, m-1$) appears for $A_{m,n}$ with $m > 2$.

In order to prove Theorem 1, we have to calculate the Carathéodory-Reiffen metric $\gamma_{A_{m,n}}$ outside of the origin.

First, let us recall its definition

$$\gamma_{A_{m,n}}((z,w); X) := \max\{|f'(z,w)X| : f \in \mathcal{O}(A_{m,n}, \mathbb{D})\},$$

where $(z,w) \in A_{m,n}$ and X a tangent vector in (z,w) at $A_{m,n}$. Recall that if $(z,w) = \zeta = p_{m,n}(\lambda)$, $\lambda \in \mathbb{D} \setminus \{0\}$, then the tangent space $T_\zeta(A_{m,n})$ at ζ is spanned by the vector $p'_{m,n}(\lambda)$. The same holds if $m = 1$ and $\lambda = 0$ whereas $T_0(A_{m,n}) = \mathbb{C}^2$ if $m \geq 2$.

Using the above description of $\mathcal{O}(A_{m,n}, \mathbb{D})$ we may reformulate this definition in the following appropriate form which will be used here:

$$\gamma_{A_{m,n}}(p_{m,n}(\lambda); p'_{m,n}(\lambda)) = \sup\{\frac{|h'(\lambda)|}{1 - |h(\lambda)|^2} : h \in \mathcal{O}_{m,n}(\mathbb{D})\}.$$

Then we have the following result.

Theorem 3. *Let* $\lambda \in \mathbb{D}$. *Then*

$$\gamma_{A_{m,n}}(p_{m,n}(\lambda); p'_{m,n}(\lambda)) = \frac{m|\lambda|^{m-1}}{1 - |\lambda|^{2m}}.$$

It follows from the results above (as in the case of domains in $\mathbb{C}^n$) that $\gamma_{A_{m,n}}$ is the infinitesimal form of $c_{A_{m,n}}$ outside the origin. More precisely, if $\lambda \in \mathbb{D} \setminus \{0\}$, then

$$\lim_{\mu \to \lambda} \frac{c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))}{|\lambda - \mu|} = \lim_{\mu \to \lambda} \frac{c_{\mathbb{D}}(\lambda^m, \mu^m)}{|\lambda - \mu|}$$
$$= \frac{m|\lambda|^{m-1}}{1 - |\lambda|^{2m}} = \gamma_{A_{m,n}}(p_{m,n}(\lambda); p'_{m,n}(\lambda)).$$

Observe that the same holds if $m = 1$ and $\lambda = 0$.

On the other hand, note that

$$\gamma_{A_{m,n}}(0; X) = \max\{|f'(z,w)X| : f \in \mathcal{O}(A_{m,n}, \mathbb{D}), f(0) = 0\}.$$

Then for such f we have $f \circ p_{m,n}(\zeta) = \zeta^m h(\zeta), \zeta \in \mathbb{D}$, where $h \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$. Observe that $\dfrac{\partial f}{\partial z}(0) = \dfrac{h^{(n-m)}(0)}{(n-m)!}$ and $\dfrac{\partial f}{\partial w}(0) = h(0)$ for $m \geq 2$. Thus, if $X = (X_1, X_2) \in \mathbb{C}^2$, then

$$\gamma_{A_{m,n}}(0; X) = \max\{|X_1 \frac{h^{(n)}(0)}{n!} + X_2 \frac{h^{(m)}(0)}{m!}| : h \in \mathcal{O}_{m,n}(\mathbb{D}), h(0) = 0\}$$

$$= \max\{|X_1 \frac{h^{(n-m)}(0)}{(n-m)!} + X_2 h(0)| : h \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}}), h^{(j)}(0) = 0, j+m \in S_{m,n}\};$$

in particular, $\gamma_{A_{m,n}}(0; X) = \|X\|$ if $X_1 X_2 = 0$. Using the first equality from above, we shall prove the following infinitesimal result at the origin.

Proposition 4. *Let* $X_{\lambda,\mu} := (\lambda^n - \mu^n, \lambda^m - \mu^m)$. *Then*

$$\lim_{\lambda,\mu \to 0, \lambda \neq \mu} \frac{c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))}{\gamma_{A_{m,n}}(0; X_{\lambda,\mu})} = 1.$$

Corollary 5. *Let* $m > 1$. *Then there are points* $\lambda, \mu \in \mathbb{D}$ *such that*

$$c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))(\lambda, \mu) > \max\{\rho(\lambda^m, \mu^m), \rho(\lambda^{m+1}, \mu^{m+1})\}.$$

It turns out that the general calculation of the Carathéodory-Reiffen metric at the origin becomes much more difficult. The next theorem may give some flavor of the nature of this formulas.

Proposition 6. *Let* $X = (X_1, X_2) \in \mathbb{C}^2$. *Then*

$$\gamma_{A_{3,4}}(0; X) = \begin{cases} |X_1| & \text{if } |X_1| \geq 2|X_2| \\ |X_2| & \text{if } |X_2| \geq \sqrt{2}|X_1| \\ |X_1| \frac{c^3 - 18c + (c^2+24)^{3/2}}{108} & \text{if } 1 < c := 2\frac{|X_2|}{|X_1|} < 2\sqrt{2} \end{cases}.$$

It seems rather difficult to calculate an effective formula of the Carathéodory distance of $A_{m,n}$. However, we have its value at pairs of "opposite" points; to be more precise the following is true.

Proposition 7. *Let* $\lambda \in \mathbb{D}$, $\lambda \neq 0$. *Then*

$$m_{A_{2,2k+1}}(p_{2,2k+1}(\lambda), p_{2,2k+1}(-\lambda)) = \frac{2|\lambda|^{2k+1}}{1 + |\lambda|^{4k+2}}.$$

Observe that now, opposite to the cases before, the number $n = 2k + 1$ appears in the formula.

Finally, the discussion of the Kobayashi distance and the Kobayashi-Royden metric on $A_{m,n}$ becomes comparably much simpler. Let us first recall the definitions of the Lempert function $\widetilde{k}_{A_{m,n}}$, the Kobayashi distance $k_{A_{m,n}}$ and the Kobayashi-Royden metric $\kappa_{A_{m,n}}$.

- $\widetilde{k}_{A_{m,n}}(\zeta,\eta) := \inf\{\rho(\lambda,\mu) : \lambda,\mu \in \mathbb{D}\ \exists_{\varphi\in\mathcal{O}(\mathbb{D},A_{m,n})} : \varphi(\lambda) = \zeta,\ \varphi(\mu)=\eta\},\quad \zeta,\eta\in A_{m,n}$;
- $k_{A_{m,n}}$:= the largest distance on $A_{m,n}$ below of $\widetilde{k}_{A_{m,n}}$;
- $\kappa_{A_{m,n}}(\zeta;X) := \inf\{\alpha\in\mathbb{R}_+ : \exists_{\varphi\in\mathcal{O}(\mathbb{D},A_{m,n})} : \varphi(0)=\zeta,\ \alpha\varphi'(0) = X\},\quad \zeta\in A_{m,n},\ X\in T_\zeta(A_{m,n})$.

We set $\widetilde{k}_{A_{m,n}}(\zeta,\eta) := \infty$ or $\kappa_{A_{m,n}}(\zeta;X) := \infty$ if there are no respective discs φ.

Since $\mathcal{O}(\mathbb{D},A_{m,n}) = \{p_{m,n}\circ\psi : \psi\in\mathcal{O}(\mathbb{D},\mathbb{D})\}$, then we have the following formulas (see also [3, 4]).

Proposition 8. *Let $\lambda,\mu\in\mathbb{D}$. Then*

$$k_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu)) = \widetilde{k}_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu)) = \rho(\lambda,\mu).$$

If $\lambda\neq 0$, then $\kappa_{A_{m,n}}(p_{m,n}(\lambda);p'_{m,n}(\lambda)) = \dfrac{1}{1-|\lambda|^2}$.

Let $X=(X_1,X_2)\in T_0A_{m,n}\setminus\{0\}$. Then

$$\kappa_{A_{m,n}}(0;X) = \begin{cases} |X_2| & \text{if } m=1 \\ \infty & \text{if otherwise}\end{cases}.$$

At the end of the paper a simple reducible variety is also discussed.

2. Proofs and additional remarks

We start with the proof of Theorem 3 which will serve as the basic information for Theorem 1.

Proof of Theorem 3. Recall that

$$\gamma_{A_{m,n}}(p_{m,n}(\lambda);p'_{m,n}(\lambda)) = \max\{\frac{|h'(\lambda)|}{1-|h(\lambda)|^2} : h\in\mathcal{O}_{m,n}(\mathbb{D})\}.$$

Observe that if $\alpha\in\mathbb{D}$ and $\Phi_\alpha(\zeta) = \frac{\alpha-\zeta}{1-\overline{\alpha}\zeta}$, then $h_\alpha = \Phi_\alpha\circ h\in\mathcal{O}_{m,n}(\mathbb{D})$ (use, for example, the Faà di Bruno formula) and

$$\frac{|h'_\alpha(\lambda)|}{1-|h_\alpha(\lambda)|^2} = \frac{|h'(\lambda)|}{1-|h(\lambda)|^2}.$$

Then

$$\gamma_{A_{m,n}}(p_{m,n}(\lambda);p'_{m,n}(\lambda))=\max\{\frac{|h'(\lambda)|}{1-|h(\lambda)|^2}:h\in\mathcal{O}_{m,n}(\mathbb{D}),h(0)=0\}$$

$$=\max\{\frac{|(\lambda^m\widetilde{h}(\lambda))'|}{1-|\lambda^m\widetilde{h}(\lambda)|^2}:\widetilde{h}\in\mathcal{O}(\mathbb{D},\overline{\mathbb{D}}),\widetilde{h}^{(j)}(0)=0,j+m\in S_{m,n}\}$$

$$=|\lambda|^{m-1}\max\{\frac{|mh(\lambda)+\lambda h'(\lambda)|}{1-|\lambda^m h(\lambda)|^2}:$$

$$h\in\mathcal{O}(\mathbb{D},\overline{\mathbb{D}}),h^{(j)}(0)=0,j+m\in S_{m,n}\}=\frac{m|\lambda|^{m-1}}{1-|\lambda|^{2m}}.$$

The last equality is a consequence of the fact that the unimodular constants are the only extremal functions for

$$\max\{\frac{|mh(\lambda)+\lambda h'(\lambda)|}{1-|\lambda^m h(\lambda)|^2}:h\in\mathcal{O}(\mathbb{D},\overline{\mathbb{D}})\}.$$

To prove this fact, observe that $(h(\lambda),h'(\lambda))$ varies on all pairs (a,b) satisfying $|b|\le\frac{1-|a|^2}{1-|\lambda|^2}$. Thus, we have to show that if $0\le c,s<1$ and $0\le t\le t_s:=\frac{1-s^2}{1-c^2}$, then $F(s,t)<F(1,0)$, where $F(s,t)=\frac{ms+ct}{1-c^{2m}s^2}$. Since $F(s,t)\le F(s,t_s)$, the problem may be reduced to the inequality

$$\frac{m(1-c^2)s+c(1-s^2)}{1-c^{2m}s^2}<\frac{m(1-c^2)}{1-c^{2m}}\iff\frac{c(1-c^{2m})}{m(1-c^2)}<\frac{1+c^{2m}s}{1+s}.$$

Using the inequality $\frac{1+c^{2m}}{2}<\frac{1+c^{2m}s}{1+s}$, one has to see that

$$\frac{c(1-c^{2m})}{m(1-c^2)}<\frac{1+c^{2m}}{2}\iff 2c\sum_{j=0}^{m-1}c^{2j}<m(1+c^{2m}).$$

Finally, by summing up the inequalities $1-c^{2j+1}>c^{2m-2j-1}(1-c^{2j+1})$ for $j=0,\dots,m-1$, the last inequality follows. □

Now, we are in the position to prove Theorem 1.

Proof of Theorem 1. Set $\Lambda_{\lambda,m}=\{\zeta\in\mathbb{D}:\operatorname{Re}(\lambda\overline{\zeta})\ge\cos(\pi/m)|\lambda\zeta|\}$, $\lambda\in\mathbb{D}$, $m\in\mathbb{N}$. Recall again that $\Lambda_{\lambda,m}$ is an angular sector around λ.

In a first step we shall prove that if $\lambda\in\mathbb{D}$ and $\mu\in\Lambda_{\lambda,m}$, then

$$c^i_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))=c_{\mathbb{D}}(\lambda^m,\mu^m).$$

Since

$$(1)\quad c^i_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))\ge c_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))\ge c_{\mathbb{D}}(\lambda^m,\mu^m),$$

we have only to prove the opposite inequality. After rotation, we may assume that $\lambda\in[0,1)$. By continuity, we may also assume that $\lambda,\mu\neq 0$ and $\arg(\mu)\in(-\pi/m,\pi/m)$. Then the geodesic for $c^i_{\mathbb{D}}(\lambda^m,\mu^m)$ does not intersect the segment $(-1,0]$. Denote by α this geodesic and by α_m its m-th root ($\sqrt[m]{1}=1$). Observe that if $\zeta,\eta\in A^*_{m,n}:=A_{m,n}\setminus\{0\}$, then

$$c^i_{A_{m,n}}(\zeta,\eta)=\inf\{\int_0^1\gamma_{A_{m,n}}(\alpha(t);\alpha'(t))dt:\alpha:[0,1]\to A^*_{m,n}\ \text{is a } C^1\text{-curve connecting } \zeta,\eta\}$$

(see Theorem 4.2.7 in [5]).

It follows by Theorem 3 that

$$\begin{aligned}c^i_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))&\le\int_0^1\gamma_{A_{m,n}}(p_{m,n}\circ\alpha_m(t);(p_{m,n}\circ\alpha_m)'(t))dt\\&=\int_0^1\frac{m|(\alpha_m(t))|^{m-1}\alpha_m'(t)|}{1-|\alpha_m(t)|^{2m}}dt=\int_0^1\frac{|\alpha'(t)|}{1-|\alpha(t)|^2}dt\\&=c^i_{\mathbb{D}}(\lambda^m,\mu^m)=c_{\mathbb{D}}(\lambda^m,\mu^m).\end{aligned}$$

It remains to prove that if $\mu\notin\varLambda_{\alpha,m}$, then

$$c^i_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))=c^i_{A_{m,n}}(p_{m,n}(\lambda),0)+c^i_{A_{m,n}}(0,p_{m,n}(\mu)).$$

By the triangle inequality, we only have to prove that

$$(2)\quad c^i_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))\ge c^i_{A_{m,n}}(p_{m,n}(\lambda),0)+c^i_{A_{m,n}}(0,p_{m,n}(\mu)).$$

Take an arbitrary C^1-curve $\alpha:[0,1]\to A^*_{m,n}$ with $\alpha(0)=p_{m,n}(\lambda)$ and $\alpha(1)=p_{m,n}(\mu)$. Let $t_0\in(0,1)$ be the smallest numbers such that $\alpha(t_0)\in\partial\varLambda_{\alpha,m}$. If $\alpha(t_0)=p(\lambda_0)$, then

$$\begin{aligned}&\int_0^1\gamma_{A_{m,n}}(\alpha(t);\alpha'(t))dt\\&=\int_0^{t_0}\gamma_{A_{m,n}}(\alpha(t);\alpha'(t))dt+\int_{t_0}^1\gamma_{A_{m,n}}(\alpha(t);\alpha'(t))dt\\&\ge c^i_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\lambda_0))+c^i_{A_{m,n}}(p_{m,n}(\lambda_0),p_{m,n}(\mu))\\&\ge c_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\lambda_0))+c_{A_{m,n}}(p_{m,n}(\lambda_0),p_{m,n}(\mu))\\&\ge c_{\mathbb{D}}(\lambda^m,\lambda_0^m)+c_{\mathbb{D}}(\lambda_0^m,\mu^m)\\&=c_{\mathbb{D}}(\lambda^m,0)+c_{\mathbb{D}}(0,\lambda_0^m)+c_{\mathbb{D}}(\lambda_0^m,\mu^m)\quad(\text{since }\lambda_0^m\in(-1,0)\\&\ge c_{\mathbb{D}}(\lambda^m,0)+c_{\mathbb{D}}(0,\mu^m).\end{aligned}$$

Now, (2) follows by taking the infimum over all curves under consideration. $\square$

Next, the proof of Corollary 2 will be given.

Proof of Corollary 2. (a) Follows by Theorem 1 and the inequality (1).

(b) The inequalities

$$c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \le \max\{c_{\mathbb{D}}(\lambda^m f(\lambda), \mu^m f(\mu)) : f \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})\}$$
$$\le \max\{c_{\mathbb{D}}(\lambda^m f(\lambda), 0) + c_{\mathbb{D}}(0, \mu^m + f(\mu)) : f \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})\}$$
$$\le c_{\mathbb{D}}(\lambda^m, 0) + c_{\mathbb{D}}(0, \mu^m) = c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$$

show that

$$c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$$

if and only if $\lambda^m f(\lambda)$ and $\mu^m f(\mu)$ lie on opposite rays and $|f(\lambda)| = |f(\mu)| = 1$ for some $f \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$, i.e., f is a unimodular constant, and $(\lambda\overline{\mu})^m < 0$.

The remaining part of Corollary 2 follows by the fact that $c_{\mathbb{D}}(z,0) + c_{\mathbb{D}}(0,w) = c_{\mathbb{D}}(z,w)$ if and only if $z\overline{w} \le 0$. $\square$

Remarks. (a) For $m \in \mathbb{N}$, consider the following distance on $\mathbb{D}$:

$$\rho^{(m)}(\lambda, \mu) := \max\{\rho_{\mathbb{D}}(\lambda^m h(\lambda), \mu^m h(\mu)) : h \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})\}.$$

Note that

$$\lim_{\varepsilon\to 0, \varepsilon\neq 0} \frac{\rho^{(m)}(\lambda, \lambda+\varepsilon)}{|\varepsilon|} = |\lambda|^{m-1} \max\{\frac{|mh(\lambda) + \lambda h'(\lambda)|}{1 - |\lambda^m h(\lambda)|^2} : h \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})\}$$
$$= \gamma_{A_{m,n}}(p_{m,n}(\lambda); p'_{m,n}(\lambda))$$

by the proof of Theorem 3. So it follows that the associated inner distance of $\rho^{(m)}$ equals $c^i_{A_{m,n}}(p_{m,n}(\cdot), p_{m,n}(\cdot))$. Then

$$c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \ge \rho^{(m)}(\lambda, \mu)$$
$$\ge c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \ge \rho(\lambda^m, \mu^m).$$

Moreover, the proof of Corollary 2 shows that the following conditions are equivalent:

- $c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = \rho^{(m)}(\lambda, \mu)$;
- $c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$;
- $c^i_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = \rho(\lambda^m, \mu^m)$;
- $\mathrm{Re}(\lambda\overline{\mu}) \ge \cos(\pi/m)|\lambda\mu|$ or $(\lambda\overline{\mu})^m < 0$.

As an application of these observations we obtain a simple proof (without calculations) of Lemma 14 in [6]:

If $a, b \in [0,1)$, $s \in (0,1]$ and $\theta \in [-\pi,\pi]$, then $\rho(a, be^{i\theta}) \le \rho(a^s, b^s e^{is\theta})$.

In fact, we may assume that $s \in \mathbb{Q}$. If $s = \frac{p}{q}$ $(1 \le p \le q)$, $\lambda = \sqrt[q]{a}$, $\mu = \sqrt[q]{b}e^{i\theta/q}$, then we have to prove that $\rho(\lambda^q, \mu^q) \le \rho(\lambda^p, \mu^p)$. But the angle between λ and μ does not exceed $\frac{\pi}{q} \le \frac{\pi}{p}$ and hence

$$\rho(\lambda^p, \mu^p) = \rho^{(p)}(\lambda, \mu) \ge \rho(\lambda^q, \mu^q)$$

(the last inequality holds for any $\lambda, \mu \in \mathbb{D}$ and $q \ge p$).

(b) Recall that

$$c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = \max\{\rho_{\mathbb{D}}(\lambda^m h(\lambda), \mu^m h(\mu)) : \\ h \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}}), h^{(j)}(0) = 0, j+m \in S_{m,n}\}.$$

If $m = 1$ or $(m,n) = (2,3)$, then $\rho^{(m)}(\lambda,\mu) = c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$, since $S_{1,n} = \emptyset$ and $S_{2,3} = \{1\}$.

On the other hand, if $m \ne 1$ and $m \ne n-1$, then the following conditions are equivalent:

- $\rho^{(m)}(\lambda,\mu) = \rho(\lambda^m, \mu^m)$;
- $\rho^{(m)}(\lambda,\mu) = c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$.

It is clear that the first condition implies the second one. For the converse, observe that as h varies over $\mathcal{O}(\mathbb{D},\mathbb{D})$, the pair $(h(\lambda), h(\mu))$ varies over all $(z,w) \in \mathbb{D}^2$ with $m_{\mathbb{D}}(z,w) \le m_{\mathbb{D}}(\lambda,\mu)$. Thus,

$$\rho^{(m)}(\lambda,\mu) = \max\{\rho_{\mathbb{D}}(\lambda^m z, \mu^m w) : z, w \in \mathbb{D} \text{ with} \\ m_{\mathbb{D}}(z,w) \le m_{\mathbb{D}}(\lambda,\mu) \text{ or } z = w \in \partial D.\}$$

It follows by the maximum principle for the continuous plurisubharmonic function $m_{\mathbb{D}}(\lambda^m\ \cdot, \mu^m w)$ that if $\rho^{(m)}(\lambda,\mu) = \rho_{\mathbb{D}}(\lambda^m z, \mu^m w)$, then either $z = w \in \partial D$, or $m_{\mathbb{D}}(z,w) = m_{\mathbb{D}}(\lambda,\mu)$. Assuming that $\rho^{(m)}(\lambda,\mu) \ne \rho(\lambda^m,\mu^m)$ excludes the first possibility. Then any extremal function h for $\rho^{(m)}(\lambda,\mu)$ satisfies $m_{\mathbb{D}}(h(\lambda), h(\mu)) = m_{\mathbb{D}}(\lambda,\mu)$, i.e., $h \in \mathrm{Aut}(\mathbb{D})$. Since any such function should be also extremal for $c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$, it follows that either $h^{(j)} \ne 0$ for any $j \in \mathbb{N}$, or h is a rotation. In particular, $m+1 \notin S_{m,n}$, i.e., $m = 1$ or $m = n-1$, a contradiction.

Let $m \ge 3$. Then $m+2 \notin S_{m,m+1}$ and hence h must be a rotation. Thus, the following conditions are equivalent:

- $\rho^{(m)}(\lambda,\mu) = \max\{\rho(\lambda^m,\mu^m), \rho(\lambda^{m+1},\mu^{m+1})\}$;
- $\rho^{(m)}(\lambda,\mu) = c_{A_{m,m+1}}(p_{m,n}(\lambda), p_{m,n}(\mu))$.

(c) Concerning the first condition from above, we point out that if $m > 1$, then by Corollary 5 there are points $\lambda, \mu \in \mathbb{D}$ such that

$$\begin{aligned}\rho^{(m)}(\lambda, \mu) &\geq c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))(\lambda, \mu)\\ &> \max\{\rho(\lambda^m, \mu^m), \rho(\lambda^{m+1}, \mu^{m+1})\}.\end{aligned}$$

On the other hand, $\rho^{(2m)}(\lambda, -\lambda) = \rho(\lambda^{2m+1}, -\lambda^{2m+1})$, since

$$\begin{aligned}&m_{\mathbb{D}}(\lambda^{2m}\Phi_\alpha(\lambda), \lambda^{2m}\Phi_\alpha(-\lambda))\\ &= \frac{2(1-|\alpha|^2)|\lambda|^{2m+1}}{|1+|\lambda|^{4m+2} - |\alpha|^2(|\lambda|^2 + |\lambda|^{4m}) + (1-|\lambda|^{4m})(\alpha\bar{\lambda} - \bar{\alpha}\lambda)|}\\ &\leq \frac{2(1-|\alpha|^2)|\lambda|^{2m+1}}{1+|\lambda|^{4m+2} - |\alpha|^2(|\lambda|^2 + |\lambda|^{4m})} \leq \frac{2|\lambda|^{2m+1}}{1+|\lambda|^{4m+2}},\end{aligned}$$

(use that $1 + |\lambda|^{4m+2} > |\lambda|^2 + |\lambda|^{4m}$).

Proof of Proposition 4. Observe that there is a constant $c > 0$ with:

- $c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \geq \max\{\rho(\lambda^m, \mu^m), \rho(\lambda^{m+1}, \mu^{m+1})\} \overset{near\ 0}{\geq} c|X_{\lambda,\mu}|$;
- $\gamma_{A_{m,n}}(0; X_{\lambda,\mu}) \geq c|X_{\lambda,\mu}|$;
- $|X_{\lambda,\mu}| \geq c|\lambda^k - \mu^k| \max\{|\lambda|^{k-n}, |\mu|^{k-n}|\}$ for any $k > n$.

Let now $h_{\lambda,\mu}$ be an extremal function for $c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$. Then

$$h_{\lambda,\mu}(\zeta) = \sum_{j=1}^{[n/m]} a_{j,\lambda,\mu}\zeta^{jm} + a_{n,\lambda,\mu}\zeta^n + \sum_{j>n, j\in\S_{m,n}} a_{j,\lambda,\mu}\zeta^j.$$

Since $|a_{j,\lambda,\mu}| \leq 1$, it follows that

$$|h_{\lambda,\mu}h(\lambda) - h_{\lambda,\mu}(\mu)| \leq H(\lambda, \mu) :=$$

$$|a_{m,\lambda,\mu}(\lambda^m - \mu^m) + a_{n,\lambda,\mu}(\lambda^n - \mu^n)| + \sum_{j=2}^{[n/m]} |\lambda^{jm} - \mu^{jm}| + \sum_{j=n+1}^{\infty} |\lambda^j - \mu^j|.$$

Thus,

$$\begin{aligned}1 &\leq \liminf_{\lambda,\mu\to 0, \lambda\neq\mu} \frac{H(\lambda, \mu)}{|h_{\lambda,\mu}(\lambda) - h_{\lambda,\mu}(\mu)|} = \liminf_{\lambda,\mu\to 0, \lambda\neq\mu} \frac{H(\lambda, \mu)}{m_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))}\\ &\leq \liminf_{\lambda,\mu\to 0, \lambda\neq\mu} \frac{|a_{m,\lambda,\mu}(\lambda^m - \mu^m) + a_{n,\lambda,\mu}(\lambda^n - \mu^n)|}{c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))}\\ &\quad + \liminf_{\lambda,\mu\to 0, \lambda\neq\mu} \frac{\sum_{j=2}^{[n/m]} |\lambda^{jm} - \mu^{jm}| + \sum_{j=n+1}^{\infty} |\lambda^j - \mu^j|}{c|X_{\lambda,\mu}|}\\ &= \liminf_{\lambda,\mu\to 0, \lambda\neq\mu} \frac{|a_m(\lambda^m - \mu^m) + a_n(\lambda^n - \mu^n)|}{m_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))} \leq \liminf_{\lambda,\mu\to 0, \lambda\neq\mu} \frac{\gamma_{A_{m,n}}(0; X_{\lambda,\mu})}{c_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))}\end{aligned}$$

(since

$$\gamma_{A_{m,n}}(0;X)=\max\{|X_1\frac{h^{(n)}(0)}{n!}+X_2\frac{h^{(m)}(0)}{m!}|:h\in\mathcal{O}_{m,n}(\mathbb{D}),h(0)=0\}).$$

The opposite inequality

$$\limsup_{\lambda,\mu\to 0,\lambda\neq\mu}\frac{\gamma_{A_{m,n}}(0;X_{\lambda,\mu})}{c_{A_{m,n}}(p_{m,n}(\lambda),p_{m,n}(\mu))}\leq 1$$

can be proven in a similar way and we omit the details. □

Proof of Corollary 5. Observe that for any neighborhood U of 0 we may find points $\lambda,\mu\in U$ such that $\lambda^m-\mu^m=\lambda^n-\mu^n\neq 0$. Then, by Proposition 4, it is enough to show that

$$\gamma_{A_{m,n}}(0;X_0)>1,\ \text{where } X_0:=(1,1).$$

Since $\gamma_{A_{m,n}}(0;X_0)$

$$=\max\{|\frac{h^{(n-m)}(0)}{(n-m)!}+h(0)|:h\in\mathcal{O}(\mathbb{D},\overline{\mathbb{D}}),h^{(j)}(0)=0,j+m\in S_{m,n}\}$$

and $\max\limits_{s\in S_{m,n}} s=nm-m-n$, then

$$\gamma_{A_{m,n}}(0;X_0)\geq\max\{|a+b|:(a,b)\in T_{n-m}\},$$

where T_{n-m} is the set of all pairs $(a,b)\in\mathbb{C}^2$ for which there is a function $h\in\mathcal{O}(\mathbb{D},\overline{\mathbb{D}})$ of the form $h(z)=a+bz^{n-m}+o(z^{nm-2m-n})$.

Let $k\in\mathbb{N}$ be such that $k(n-m)\geq nm-2m-n$. We shall show that there is a function $f\in\mathcal{O}(\mathbb{D},\overline{\mathbb{D}})$ of the form $f(z)=a+bz+o(z^k)$ such that $a,b>0$ and $a+b>1$, which will imply that $\gamma_{A_{m,n}}(0;X_0)>1$.

Note that by Shur's theorem (cf. [1]) such a function f exists if and only if

$$(3)\quad (1-|a|^2)X_1^2+(1-|a|^2-|b|^2)\sum_{j=2}^n X_j^2\geq 2|ab|\sum_{j=2}^n X_{j-1}X_j,\quad X\in\mathbb{R}^n.$$

Since $\cos\frac{\pi}{n+1}$ is the maximal eigenvalue of the quadratic form $\sum_{j=2}^n X_{j-1}X_j$, it follows that

$$\cos\frac{\pi}{n+1}\sum_{j=1}^n X_j^2\geq\sum_{j=2}^n X_{j-1}X_j,\quad X\in\mathbb{R}^n.$$

Then all pairs $(a,b)\in\mathbb{C}^2$ for which $2\cos\frac{\pi}{n+1}|ab|\leq 1-|a|^2-|b|^2$ satisfy (3); in particular, we may choose $a,b>0$ such that $2ab>1-a^2-b^2$, i.e., $a+b>1$. □

Now we turn to the discussion of the Carathéodory-Reiffen pseudometric on the $(3,4)$-parabola.

Proof of Proposition 6. Recall that

$$\gamma_{A_{3,4}}(0;X)=\max\{|X_1h'(0)+X_2h(0)|:h\in\mathcal{O}(\mathbb{D},\overline{\mathbb{D}}),h''(0)=0\}.$$

So, we have to describe the pairs $(a_0,a_1)\in\mathbb{C}^2$ for which there is a function $h\in\mathcal{O}(\mathbb{D},\overline{\mathbb{D}})$ of the form $h(\zeta)=a_0+a_1\zeta+\mathrm{o}(\zeta^2)$. Let I_3 be the 3×3 unit matrix and

$$M=\begin{bmatrix}a_0 & a_1 & 0\\ 0 & a_0 & a_1\\ 0 & 0 & a_0\end{bmatrix}.$$

It follows by Schur's theorem (cf. [1]) that such an h exists if only if I_3-M^*M is a semipositive matrix. It is easy to check that the last conditions just means that the pair $(|a_0|^2,|a_1|^2)$ belongs to the set

$$C:=\{(a,b)\in\mathbb{R}^2_+:a+\sqrt{b}\le 1,ab(1-a)\le((1-a)^2-b)(1-a-b)\}.$$

The second inequality can be written as

$$b\le(1-a)(1-\sqrt{a})\quad\text{ or }\quad b\ge(1-a)(1+\sqrt{a}).$$

Hence $C=\{(a,b)\in\mathbb{R}^2_+:b\le(1-a)(1-\sqrt{a}),\ a\le 1\}$. Thus,

$$\begin{aligned}\gamma_{A_{3,4}}(0;X)&=\max\{|X_1|\sqrt{b}+|X_2|\sqrt{a}:(a,b)\in C\}\\ &=\max\{t\in[0;1]:|X_1|(1-t)\sqrt{1+t}+|X_2|t\}.\end{aligned}$$

Straightforward calculations show that the last maximum is equal to

$$\begin{cases}|X_1| & \text{if } |X_1|\ge 2|X_2|\\ |X_2| & \text{if } |X_2|\ge\sqrt{2}|X_1|\\ |X_1|\frac{c^3-18c+(c^2+24)^{3/2}}{108} & \text{if } 1<c:=2\frac{|X_2|}{|X_1|}<2\sqrt{2}\end{cases}.\qquad\square$$

Now, we shall go to prove Proposition 7.

Proof of Proposition 7. Recall that

$$\begin{aligned}&m_{A_{2,2k+1}}(p_{2,2k+1}(\lambda),p_{2,2k+1}(\mu))\\ &=\max\{m_{\mathbb{D}}(f(\lambda),f(\mu)):f\in\mathcal{O}(\mathbb{D},\mathbb{D}),f^{(2j-1)}(0)=0,j=1,\dots,k\}\\ &=\max\{m_{\mathbb{D}}(\lambda^2h(\lambda),\mu^2h(\mu)):\\ &\qquad\qquad h\in\mathcal{O}(\mathbb{D},\overline{\mathbb{D}}),h^{(2j-1)}(0)=0,j=1,\dots,k-1\}.\end{aligned}$$

It follows that

$$\begin{aligned}&m_{A_{2,2k+1}}(p_{2,2k+1}(\lambda),p_{2,2k+1}(\mu))\\ &=\sup\{m_{\mathbb{D}}(\lambda^2z,\mu^2w):m_{\mathbb{D}}(z,w)\le m_{A_{2,2k-1}}(p_{2,2k-1}(\lambda),p_{2,2k-1}(\mu)).\end{aligned}$$

Then Proposition 7 will follow by induction on $n \in \mathbb{Z}_+$ if we show that

$$m_{\mathbb{D}}(z,w) \le \frac{2|\lambda|^{2k-1}}{1+|\lambda|^{4k-2}} \implies m_{\mathbb{D}}(\lambda^2 z, \lambda^2 w) \le \frac{2|\lambda|^{2k+1}}{1+|\lambda|^{4k+2}}.$$

Since $\frac{2|\lambda|^{2k-1}}{1+|\lambda|^{4k-2}} = m_{\mathbb{D}}(\lambda^{2k-1}, -\lambda^{2k-1})$, we may assume as in Remark (b) that $z = \Phi_\alpha(\lambda^{2k-1})$ and $w = \Phi_\alpha(-\lambda^{2k-1})$ for some $\alpha \in \mathbb{D}$. Then

$$\begin{aligned} &m_{\mathbb{D}}(\lambda^2 z, \lambda^2 w) \\ &= \frac{2(1-|\alpha|^2)|\lambda|^{2k+1}}{|1+|\lambda|^{4k+2} - |\alpha|^2(|\lambda|^4 + |\lambda|^{4k-2}) + (1-|\lambda|^4)(\alpha\bar{\lambda}^{2k-1} - \bar{\alpha}\lambda^{2k-1})|} \\ &\le \frac{2(1-|\alpha|^2)|\lambda|^{2k+1}}{1+|\lambda|^{4k+2} - |\alpha|^2(|\lambda|^4 + |\lambda|^{4k-2})} \le \frac{2|\lambda|^{2k+1}}{1+|\lambda|^{4k+2}}, \end{aligned}$$

since $1+|\lambda|^{4k+2} > |\lambda|^4 + |\lambda|^{4k-2}$. □

Remark. From the result above one may conclude the following interpolation result. Namely, for given $k \in \mathbb{N}$, $\lambda, \eta, \zeta \in \mathbb{D}$ the following conditions are equivalent:

(i) $m_{\mathbb{D}}(\eta, \zeta) \le m_{\mathbb{D}}(\lambda^{2k+1}, -\lambda^{2k-1})$;
(i) $\exists_{f \in \mathcal{O}(\mathbb{D},\mathbb{D})} : f(\lambda^{2k+1}) = \eta, f(-\lambda^{2k+1}) = \zeta$;
(iii) $\exists_{f \in \mathcal{O}(\mathbb{D},\mathbb{D})} : f(\lambda) = \eta, f(-\lambda) = \zeta, f^{(j)}(0) = 0, j = 1, \dots, 2k$;
(iv) $\exists_{f \in \mathcal{O}(\mathbb{D},\mathbb{D})} : f(\lambda) = \eta, f(-\lambda) = \zeta, f^{(2j-1)}(0) = 0, j = 1, \dots, k$.

Indeed, it is trivial that $(i) \implies (ii) \implies (iii) \implies (iv)$, and the implication $(iv) \implies (i)$ follows by the equalities

$$m_{A_{2,2k+1}}(p_{2,2k+1}(\lambda), p_{2,2k+1}(\mu)) = \frac{2\lambda|^{2k+1}}{1+|\lambda|^{4k+2}} = m_{\mathbb{D}}(\lambda^{2k+1}, -\lambda^{2k-1}).$$

Finally, we discuss the proof for the Kobayashi distance and metric.

Proof of Proposition 8. The proof of the formula for $\widetilde{k}_{A_{m,n}}$ follows the one for the case $(m,n) = (2,3)$ (see [4]). For convenience of the reader we include it.

First, $\widetilde{k}_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) \le \rho(\lambda, \mu)$ because $p_{m,n}$ is holomorphic. Second, since m and n are relatively prime, it is easy to see that $\mathcal{O}(\mathbb{D}, A_{m,n}) = \{p_{m,n} \circ \psi : \psi \in \mathcal{O}(\mathbb{D},\mathbb{D})\}$. Then any $\varphi \in \mathcal{O}(\mathbb{D}, A_{m,n})$ with $\varphi(\widetilde{\lambda}) = p_{m,n}(\lambda)$ and $\varphi(\widetilde{\mu}) = p_{m,n}(\mu)$ corresponds to some $\psi \in \mathcal{O}(\mathbb{D},\mathbb{D})$ with $\psi(\widetilde{\lambda}) = \lambda$ and $\psi(\widetilde{\mu}) = \mu$. Thus, $\rho(\lambda,\mu) \le \rho(\widetilde{\lambda}, \widetilde{\mu})$ and hence $\rho(\lambda,\mu) \le \widetilde{k}_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu))$. So, $\widetilde{k}_{A_{m,n}}(p_{m,n}(\lambda), p_{m,n}(\mu)) = \rho(\lambda,\mu)$; in particular, $\widetilde{k}_{A_{m,n}}$ is a distance and therefore $\widetilde{k}_{A_{m,n}} = k_{A_{m,n}}$.

The formulas for $\kappa_{A_{m,n}}$ can be proven in a similar way and we omit the details. □

We conclude this paper by mentioning the simplest example of a reducible variety.

Remark. Put $A_{2,2} := \{(z,w) \in \mathbb{D}^2 : z^2 = w^2\}$; $A_{2,2}$ is reducible. Obviously, $A_{2,2}$ is biholomorphically equivalent to the coordinate cross $V := \{(z,w) \in \mathbb{D}^2 : zw = 0\}$. Therefore, we discuss V instead of $A_{2,2}$.

It is clear that $c_V((z_1,0),(z_2,0)) = \tilde{k}_V((z_1,0),(z_2,0)) = \rho(z_1,z_2)$,

$$\tilde{k}_V((z,0),(0,w)) = \infty \ (zw \neq 0)$$

and

$$k_V((z,0),(0,w)) = \tilde{k}_V((z,0),(0,0)) + \tilde{k}_V((0,0),(0,w)) = \rho(|z|,-|w|).$$

Moreover, $\gamma_V((z,0);(1,0)) = \kappa_V((z,0);(1,0)) = \dfrac{1}{1-|z|^2}$ and

$$\kappa_V(0;X) = \begin{cases} |X| & \text{if } X_1X_2 = 0 \\ \infty & \text{if otherwise} \end{cases}.$$

Recall now that

$$\mathcal{O}(V,\mathbb{D}) = \{f + g - f(0) : \\ f \in \mathcal{O}(\mathbb{D}\times\{0\},\mathbb{D}), g \in \mathcal{O}(\{0\}\times\mathbb{D},\mathbb{D}), f(0) = g(0)\}.$$

Then obviously $\gamma_V(0;X) = |X_1| + |X_2|$.

Finally, since $z + w \in \mathcal{O}(V,\mathbb{D})$, it follows that

$$c_V((z,0),(0,w)) = c_V((|z|,0),(-|w|,0)) \geq \rho(|z|,-|w|).$$

Thus, $c_V = k_V$; in particular, $c_V = c_V^i$.

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